Int J Theor Phys (2009) 48: 3614–3622
DOI 10.1007/s10773-009-0168-x

Hawking Temperature Calculation Using Tunneling Mechanism

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Received: 7 August 2009 / Accepted: 7 October 2009 / Published online: 15 October 2009

Abstract Based on the black hole tunneling mechanism in which the ingoing particles should be absorbed absolutely, Hamilton-Jacobi method is modified to investigate Hawking radiation from a Schwarzschild black hole once again. The results are independent on the coordinates and the standard Hawking temperature is obtained without the factor of 2 problem. Moreover, it is also pointed out that the modified Hamilton-Jacobi method can be applied to investigate Hawking radiation from the general horizons in dynamical spacetimes.

Keywords Hawking radiation · Hamilton-Jacobi method · Quantum tunneling · Black hole

1 Introduction

One of the most important theoretical discoveries in black hole physics obtained by Stephen Hawking in 1970's is the realization that a black hole is not really black and it can radiate thermally like a black body [1, 2]. Ever since Hawking's original derivation of Hawking radiation from a Schwarzschild black hole under quantum field theory in a curved background, there are several methods for deriving Hawking radiation and calculating its temperature. Recently, a quasi-classical method of taking Hawking radiation as a tunneling effect was proposed [3–11]. In the model, black hole radiation can be described by pair creation near the black hole horizon with the subsequent tunneling process: as a virtual pair of a particle with negative energy inside and a particle with positive energy outside the horizon appears, it can materialize into a real pair with zero total energy. Indeed the tunneling process can be analogous to the Schwinger mechanism for pair production by an external electric field [12]. One approach uses Hamilton's equation on null geodesics to calculate Hawking temperature

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during the tunneling process [3–7]. Another one uses Hamilton-Jacobi equation to obtain Hawking temperature [8–11].

However, in the second Hamilton-Jacobi method, one gave the temperature twice as large as the correct one in the case of a black hole with standard Schwarzschild-like metric [13–15]. In order to solve the factor of 2 problem, there are two methods: one [16] is based on the detailed balancing formula that was first proposed in Ref. [17], the other one [18–21] is based on the canonical invariance formula. Both of them are modifying the imaginary contribution to the classical action. However, the physics mechanism interpretation are all ambiguous. So it is necessary to find out another method which can not only obtain correct Hawking temperature but also have a clear physical tunneling picture.

In this letter, based on the wave functions' validity to describe the tunneling process with a background of black hole, we will improve the previous methods. Considering that the ingoing particles should be absorbed absolutely, the ingoing wave function should be continuous through the horizon. In the meantime, we modify the outgoing wave function. Using WKB approximation and considering Damour–Sannan's work [22, 23], we calculate the tunneling possibility Γ and obtain correct Hawking temperature.

The organization of this paper is as follows. In Sect. 2, we will review the factor of 2 problem in tunneling process. In Sect. 3, we will discuss the wave functions' validity to describe Hawking radiation. Based on modifying the ingoing wave function and outgoing wave function, we will get correct Hawking temperature. In Sects. 4 and 5, using this new method, we will calculate Hawking temperature in Painlevé coordinates and advanced Eddington-Finkelstein coordinates respectively. Finally, we will give some discussions and conclusions in Sect. 6.

2 Factor of 2 Problem in Tunneling Formalism

Considering a Schwarzschild black hole, the standard Schwarzschild metric can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega^2, \tag{1}$$

where M is the Arnowwitt-Deser-Misner (ADM) mass of the black hole, $d\Omega^2$ is the metric on the unit two-sphere. Its event horizon is located at $r_H = 2M$.

According to quantum field theory in a curved spacetime, the action is

$$\begin{aligned} S[\phi, g_{\mu\nu}] &= \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= \frac{1}{2}\int dtdrd\theta d\varphi r^2 \sin\theta\phi\left(-\frac{1}{1-\frac{2M}{r}}\partial_t^2 + \partial_r\left(1-\frac{2M}{r}\right)\partial_r + \frac{1}{r^2}\triangle_\Omega\right)\phi. \end{aligned} \tag{2}$$

Now using the limit $r \to r_H = 2M$ and leaving only dominant terms, the action becomes

$$S[\phi, g_{\mu\nu}] = \sum_{n,l}\frac{1}{2}\int r^2 dtdr\phi_{nl}\left(-\frac{1}{1-\frac{2M}{r}}\partial_t^2 + \partial_r\left(1-\frac{2M}{r}\right)\partial_r\right)\phi_{nl}, \tag{3}$$

where $\phi = \sum_{n,l,m}\phi_{nl}Y_{lm}$ is expanded by 2-dimensional spherical harmonics. Thus this action is infinite set of the scalar fields on the 2-dimensional metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})}, \tag{4}$$

so the action of the outgoing massless particle is given by the classical action S that satisfies the relativistic Hamilton-Jacobi action

$$g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) = -\frac{1}{1-\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)^2 + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 = 0, \tag{5}$$

where $g^{\mu\nu}$ is the inverse metric of the black ground space-time, S is the action of the particle. The corresponding scalar field can be written as

$$\psi \propto \exp[-iS]. \tag{6}$$

For stationary space time with a time-like Killing vector, we can split the action into a time and spatial part as

$$S = \omega t + S_0(r). \tag{7}$$

Putting (7) into (5), we have

$$S(r) = \omega t \pm \omega \int \frac{dr}{1-\frac{2M}{r}}. \tag{8}$$

The rate for a quasi-classical tunneling process can be written as

$$\Gamma \propto \exp[-2\mathrm{Im}S(r)] = \exp[-2\mathrm{Im}S_0(r)]. \tag{9}$$

Thinking of Boltzmann factor $\Gamma \propto \exp(-\beta\omega) = \exp(-\frac{\omega}{T})$, we can get a temperature as

$$T = \frac{1}{4\pi M} = 2T_H, \tag{10}$$

which is twice of correct Hawking temperature.

3 An Improved Method for Tunneling Mechanism

Why does the above calculation give a wrong result? We will propose a new method to investigate Hawking radiation based on tunneling mechanism, which can avoid this problem.

Substituting (8) into (6), we have the wave function as

$$\psi \propto \exp\left\{-i\omega\left[t \pm \int \frac{dr}{1-\frac{2M}{r}}\right]\right\}, \tag{11}$$

which contains the ingoing wave and outgoing wave respectively

$$\psi_{in} \propto \exp\left\{-i\omega\left[t - \int \frac{dr}{1-\frac{2M}{r}}\right]\right\}, \quad \psi_{out} \propto \exp\left\{-i\omega\left[t + \int \frac{dr}{1-\frac{2M}{r}}\right]\right\}. \tag{12}$$

Both of them have singularity at the horizon $r = 2M$. However, considering the character of a classical black hole, ingoing particles should be completely absorbed. That is to say, the ingoing wave should be continuous function through the horizon. So the upper ingoing wave function can not describe the ingoing wave of the black hole. In order to obtain a perfect wave function, we consider the wave function outside the horizon firstly.

When $r > 2M$, integrating (8), we have

$$S(r) = \omega t \pm \omega\left(r + 2M \ln \frac{r-2M}{2M} + C\right), \tag{13}$$

so (11) should be

$$\psi(r) \propto \exp\left\{-i\omega\left[t \pm \left(r + 2M \ln \frac{r-2M}{2M} + C\right)\right]\right\}. \tag{14}$$

In the region $r > 2M$, wave function can be obtained as

$$\begin{aligned} \psi_{in}(r > 2M) &\propto \exp\left\{-i\omega\left[t + \left(r + 2M \ln \frac{r-2M}{2M} + C\right)\right]\right\}, \\ \psi_{out}(r > 2M) &\propto \exp\left\{-i\omega\left[t - \left(r + 2M \ln \frac{r-2M}{2M} + C\right)\right]\right\}. \end{aligned} \tag{15}$$

For further discussions, it is convenient to introduce the set of null tortoise coordinates which are defined as

$$v = t + r + 2M \ln \frac{r-2M}{2M} + C, \tag{16}$$

so we have outgoing wave and ingoing wave respectively

$$\psi_{out} \propto e^{-i\omega v} e^{2iw(r+C)} e^{4iMw \ln \frac{r-2M}{2M}} = e^{-i\omega v} e^{2iw(r+C)} \left(\frac{r-2M}{2M}\right)^{4iM\omega}, \quad \psi_{in} \propto e^{-i\omega v}. \tag{17}$$

In (17), we get continuous ingoing wave which has no singularity at the horizon and its determinant is nonzero. Now ingoing wave function can exactly reflect the characteristic of the black hole ingoing particles. Meanwhile, the outgoing wave function can only describe the outgoing particles' behavior outside the horizon. In order to extend the outgoing wave function inside the horizon, letting $(r - 2M) \rightarrow | r - 2M | e^{-i\pi} = (2M - r)e^{-i\pi}$, so the outgoing wave function inside the horizon can be obtained as

$$\psi_{out}(r < 2M) = e^{-i\omega v} e^{2iw(r+C)} \left(\frac{2M-r}{2M}\right)^{4iM\omega} e^{4\pi M\omega}. \tag{18}$$

Using Heaviside function Θ, we can write the total outgoing wave function as

$$\psi_{\omega out} = N_\omega \exp\{-i[\Theta(r-2M)\psi_{out}(r > 2M) + \Theta(2M-r)\psi_{out}(r < 2M)]\}. \tag{19}$$

As $\psi_{\omega out}$ was already normalized

$$(\psi_{\omega out}, \psi_{\omega out}) = \pm 1, \tag{20}$$

where "+" and "−" corresponds bosons and fermions respectively, so we get

$$N_\omega^2 = \frac{1}{e^{8\pi M\omega} \pm 1}. \tag{21}$$

Therefore, according to Damour–Sannan's work in Ref. [4], the emission rate of the outgoing wave at the horizon can be given

$$\Gamma = \frac{|\psi_{out}(r > 2M)|^2}{|\psi_{out}(r < 2M)|^2} = e^{-8\pi M\omega}, \tag{22}$$

and the corresponding thermal temperature is

$$T = \frac{1}{8\pi M}, \tag{23}$$

which is just the standard Hawking temperature.

4 Hawking Temperature in Painlevé Coordinates

In this section, we will apply the above method to investigate Hawking radiation form a black hole in Painlevé coordinates. We can express it as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2. \tag{24}$$

Near the horizon $r = 2M$, this metric can also be reduced to 2-dimensional metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2. \tag{25}$$

The relativistic Hamilton-Jacobi equation of massless particles is

$$-\left(\frac{\partial S}{\partial t}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + 2\sqrt{\frac{2M}{r}}\left(\frac{\partial S}{\partial t}\right)\left(\frac{\partial S}{\partial r}\right) = 0. \tag{26}$$

Using (7), we obtain

$$S_0(r) = -\omega \int \frac{dr}{1 - \frac{2M}{r}}\sqrt{\frac{2M}{r}} \pm \omega \int \frac{dr}{1 - \frac{2M}{r}}, \tag{27}$$

so the wave function is

$$\psi \propto \exp\left\{-i\omega\left[t - \omega \int \frac{dr}{1 - \frac{2M}{r}}\sqrt{\frac{2M}{r}} \pm \omega \int \frac{dr}{1 - \frac{2M}{r}}\right]\right\}. \tag{28}$$

Apparently, the ingoing and outgoing wave can be respectively expressed as

$$\psi_{in} \propto \exp\left\{-i\omega\left[t - \omega \int \frac{dr}{1 - \frac{2M}{r}}\sqrt{\frac{2M}{r}} + \omega \int \frac{dr}{1 - \frac{2M}{r}}\right]\right\},$$

$$\psi_{out} \propto \exp\left\{-i\omega\left[t - \omega \int \frac{dr}{1 - \frac{2M}{r}}\sqrt{\frac{2M}{r}} - \omega \int \frac{dr}{1 - \frac{2M}{r}}\right]\right\}. \tag{29}$$

Due to the ingoing wave function has singularity at the horizon, it can not describe the ingoing particles in the black hole. In order to obtain a perfect ingoing wave function, we introduce the set of new coordinates which are defined as

$$v = t - \omega \int \frac{dr}{1-\frac{2M}{r}} \sqrt{\frac{2M}{r}} + \omega \int \frac{dr}{1-\frac{2M}{r}}, \tag{30}$$

so we express ingoing and outgoing waves as

$$\psi_{in} \propto e^{-i\omega v}, \quad \psi_{out} \propto e^{-i\omega v} e^{2i\omega \int \frac{dr}{1-\frac{2M}{r}}} = e^{-i\omega v} e^{2i\omega(r+C)} \left(\frac{r-2M}{2M} \right)^{4iM\omega}. \tag{31}$$

The ingoing wave does not have singularity at the horizon, and outgoing wave is only applicable for the region outside the horizon, so we extend outgoing wave inside the horizon. Letting $(r-2M) \rightarrow \mid r-2M \mid e^{-i\pi} = (2M-r)e^{-i\pi}$, the outgoing wave function inside the horizon can be obtained as

$$\psi_{out}(r < 2M) = e^{-i\omega v} e^{2i\omega(r+C)} \left(\frac{2M-r}{2M} \right)^{4iM\omega} e^{4\pi M\omega}. \tag{32}$$

The total normalized outgoing wave function can be rewritten as

$$\psi_{\omega out} = N_\omega \exp\{-i[\Theta(r-2M)\psi_{out}(r > 2M) + \Theta(2M-r)\psi_{out}(r < 2M)]\}, \tag{33}$$

where

$$N_\omega^2 = \frac{1}{e^{8\pi M\omega} \pm 1}. \tag{34}$$

The emission rate of the outgoing wave at the horizon is

$$\Gamma = \frac{\mid \psi_{out}(r > 2M) \mid^2}{\mid \psi_{out}(r < 2M) \mid^2} = e^{-8\pi M\omega}, \tag{35}$$

and Hawking temperature is

$$T = \frac{1}{8\pi M} = T_H. \tag{36}$$

5 Hawking Radiation in Advanced Eddington-Finkelstein Coordinates

The metric of a Schwarzschild black hole in advanced Eddington-Finkelstein coordinates reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{37}$$

where v is the advanced Eddington-Finkelstein time. Obviously, Hamilton-Jacobi equation is

$$2\frac{\partial S}{\partial v}\frac{\partial S}{\partial r} + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 = 0. \tag{38}$$

Using (7), we can obtain two solutions

$$S_{out} = \omega v - \omega \int \frac{2dr}{1-\frac{2M}{r}}, \qquad S_{in} = \omega v. \tag{39}$$

The wave functions can be written as

$$\psi_{in} \propto \exp\left\{-i\omega\left[v - \int \frac{2dr}{1-\frac{2M}{r}}\right]\right\}, \qquad \psi_{out} \propto \exp\{-i\omega v\}. \tag{40}$$

The ingoing wave function is continuous, we can use this wave to describe the behavior of ingoing particles. Extending the outgoing wave inside the horizon, we have the total normalized outgoing wave function as

$$\psi_{out}(r > 2M) \propto \exp\left\{-i\omega\left[\omega v - \omega \int \frac{2dr}{1-\frac{2M}{r}}\right]\right\} = e^{-i\omega v} e^{2i\omega(r+C)}\left(\frac{r-2M}{2M}\right)^{4iM\omega},$$

$$\psi_{out}(r < 2M) \propto e^{-i\omega v} e^{2i\omega(r+C)}\left(\frac{2M-r}{2M}\right)^{4iM\omega} e^{4\pi M\omega}, \tag{41}$$

or

$$\psi_{\omega out} = N_\omega \exp\{-i[\Theta(r-2M)\psi_{out}(r > 2M) + \Theta(2M-r)\psi_{out}(r < 2M)]\}, \tag{42}$$

where

$$N_\omega^2 = \frac{1}{e^{8\pi M\omega} \pm 1}. \tag{43}$$

The emission rate of the outgoing wave at the horizon is obtained as

$$\Gamma = \frac{|\psi_{out}(r > 2M)|^2}{|\psi_{out}(r < 2M)|^2} = e^{-8\pi M\omega}, \tag{44}$$

and Hawking temperature is

$$T = \frac{1}{8\pi M} = T_H. \tag{45}$$

6 Discussions and Conclusions

To obtain correct standard Hawking temperature using Hamilton-Jacobi method, the wave function should be applicable to the special tunneling mechanism of the black hole. As a black hole should absorb the ingoing particles completely, the ingoing wave function should be continuous through the horizon. The calculation in Sect. 2 has neglected that the ingoing wave function has singularity at the horizon and has directly applied the formula of the quasi-classical tunneling process, so wrong result is obtained. Considering the wave functions according to (6) and (13), one can get the outgoing wave function in (15) as

$$\psi_{out}(r > 2M) \propto \exp\left\{-i\omega\left[t - \left(r + 2M\ln\frac{r-2M}{2M} + C\right)\right]\right\}$$

$$= e^{-i\omega t} e^{i\omega(r+C)} \left(\frac{r-2M}{2M}\right)^{2iM\omega},$$

$$\psi_{out}(r < 2M) \propto e^{-i\omega t} e^{i\omega(r+C)} \left(\frac{r-2M}{2M}\right)^{2iM\omega} e^{2\pi M\omega}, \tag{46}$$

or

$$\psi_{\omega out} = N_\omega \exp\{-i[\Theta(r-2M)\psi_{out}(r > 2M) + \Theta(2M-r)\psi_{out}(r < 2M)]\}, \tag{47}$$

where

$$N_\omega^2 = \frac{1}{e^{4\pi M\omega} \pm 1}. \tag{48}$$

Thus the emission rate of the outgoing wave at the horizon is

$$\Gamma = \frac{|\psi_{out}(r > 2M)|^2}{|\psi_{out}(r < 2M)|^2} = e^{-4\pi M\omega}, \tag{49}$$

and the corresponding temperature is

$$T = \frac{1}{4\pi M}. \tag{50}$$

It is obvious that these wave functions neglecting black hole model limitation give the wrong Hawking temperature which is twice as correct Hawking temperature. In Sect. 3, after considering this limitation, we have improved Hamilton-Jacobi method to give the correct Hawking temperature. We would like to point out that the key problem of the factor of 2 problem is that the ingoing wave function has singularity at the horizon and these wave functions can not describe the tunneling process of Hawking radiation.

Using this method, the temperature of a Schwarzschild black hole in Painlevé coordinates and advanced Eddington-Finkelstein coordinates is also investigated. The conclusion can be consistent with the standard Hawking temperature. The results are independent on the coordinates. In the meantime, it is found that the wave functions are naturally applicable to describe the tunneling mechanism of the black hole Hawking radiation in the advanced Eddington-Finkelstein coordinates.

Moreover, thinking of the black hole tunneling mechanism, this improved Hamilton-Jacobi method in which the ingoing particles should be absorbed absolutely, can be applied to investigate Hawking radiation from the general horizons in dynamical spacetimes. The thermodynamical equilibrium at the horizon is not necessary, and this is more general method than Ref. [16, 18–21].

Acknowledgements One of the authors Xianming Liu would like to give great thanks for the helpful discussions with Prof. Zheng Zhao, Dr. Shiwei Zhou and Bo Liu. This work is supported by the National Natural Science Foundation of China (Grant No. 10773002, 10875012) and the National Basic Research Program of China (Grant No. 2003CB716302).

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